Abstract Algebra IV
Mid-Semester Exam

Note: You may use any result proved in class, *unless* the question is asking you to (re-)prove a result that we have already proved in class! On the other hand, if you need to use the result of a *homework problem* to answer a question below, then you must put down the solution to that homework problem also (i.e., you cannot simply quote the result of a homework problem).

Note: Each problem is worth the same amount.

(1) Show that the only field automorphism of $\mathbb{R}$ is the identity.

(2) State carefully the Isomorphism Extension Theorem. (Statement only, no proof.) Now use this theorem to determine the Galois group of the splitting field of $x^3 - 5$ over $\mathbb{Q}$. (You must justify why your maps are valid field automorphisms. This is where the Isomorphism Extension Theorem comes in .)

(3) (We did most of this problem in class!) Let K/F and L/F be finite Galois extensions.
  (a) Show that KL/F is Galois.
  (b) Show that there is an injective group homomorphism $i\colon \mathrm{Gal}(KL/F) \to \mathrm{Gal}(K/F) \times \mathrm{Gal}(L/F)$.
  (c) Determine the image $i(\mathrm{Gal}(KL/F))$ of this homomorphism (i.e., give, with proof, a description of the image subgroup).
  (d) Show that the map i is surjective if and only if $K \cap L = F$.

(4) Let K/F be a finite Galois extension, and let L be an intermediary subfield (i.e., $F \subseteq L \subseteq K$). Let $H = \mathrm{Gal}(K/L)$. Let $N(H)$ be the normailizer of H in $\mathrm{Gal}(K/F)$, i.e., $N(H) = \{g \in \mathrm{Gal}(K/F) \mid gHg^{-1} = H\}$. Let L_0 be the fixed field of $N(H)$. Show that L/L_0 is Galois, and that if E is any field with $F \subseteq E \subseteq L$ such that L/E is Galois, then $E \supseteq L_0$.

(5) Let p be a prime, and n a positive integer. This exercise will show how to construct a field extension of $\mathbb{F}_p$ of degree n. Write q for p^n.
  (a) Show that the polynomial $x^q - x$ in $\mathbb{F}_p[x]$ has no repeated roots.
  (b) Write $\mathbb{F}_q$ for the set of roots of $x^q - x$ in a fixed algebraic closure of $\mathbb{F}_p$. Show that $\mathbb{F}_p \subseteq \mathbb{F}_q$.
  (c) Show that the set F_q is a field (i.e., the sum, difference, and product of two roots of $x^q - x$ is another root of $x^q - x$ and the multiplicative inverse of a nonzero root is another root; note that by the previous part, both 0 and 1 are in $\mathbb{F}_q$ and note that commtativity, associativity, etc. come from being contained in the algebraic closure of $\mathbb{F}_p$).
  (d) Show that $|\mathbb{F}_q| = q$ and $[\mathbb{F}_q : \mathbb{F}_p] = n$.

(6) Let K/F be a separable extension of degree p^2, where p is a prime. Write $K = F(a)$ for suitable $a \in K$. If K contains more than p roots of the minimal polynomial of a over F, show that K is normal. Show that there are only two possible structures for $Gal(K/F)$.